Mathematics A level

Course plan

This plan shows the structure of the course and gives an outline of the contents. Sections 1–5 cover the requirements of the AS and Part 1 of the A level; Sections 6–10 cover Part 2 of the A level. You need to do Sections 1–10 to prepare for the A level.

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AS/A Mathematics Course guide

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Sample of the A Level Mathematics from Section 1

Topic 1.2

Indices and surds

1.2.1 Introduction

You are probably already familiar with expressions such as x^2 and x^3 and know that the numbers 2 and 3 (the **indices**, plural of **index**, or **powers**) tell you how many times you should multiply x (the **base**) by itself, i.e. $x^2 = x \times x$ and $x^3 = x \times x \times x$. In general x^m, where m is some positive integer, means x has been multiplied by itself m times. (The words **indices** and **powers** will be used interchangeably throughout this topic.)

In fact, this idea can be extended to include all kinds of numbers as indices, not just positive integers. We can, for example, have x^2, $x^{\frac{1}{2}}$, x^0, $x^{-1.78}$ and so on, and all these have their own particular meaning.

There are only a few rules for indices and they appear quite simple. This is deceptive. For many students this is a tricky topic and causes quite a bit of trouble. At the same time it is extremely important for later work. So it really is worth spending a while on this topic, making sure that you completely understand the rules and are confident that you can answer the corresponding examples.

Objectives

When you have completed this topic, you should be able to:

distinguish between situations when you should add or multiply powers

- rewrite a $\frac{1}{a^n}$ as a^{-n}
- simplify expressions involving different powers
- evaluate numbers raised to fractional powers
- solve equations involving unknown powers
- understand what we mean by a **surd**
- add and multiply surds
- rationalise the denominator when this contains surds
- simplify expressions involving surds.

1.2.2 Rules of indices

Multiplying terms with the same base

To rewrite an expression such as $a^4 \times a^3$ more simply, we can first of all expand them into their equivalent forms:
$a^4 = a \times a \times a \times a$ and $a^3 = a \times a \times a$ and then we have

$$a^4 \times a^3 = [a \times a \times a \times a] \times [a \times a \times a] = a^7$$

So *if the base is the same* and two terms are multiplied together, we simply *add* the powers

Rule 1	$x^m \times x^n = x^{m+n}$

Terms raised to a power

We can use the last rule to simplify expressions such as $(a^3)^4$. Here the base is a^3 and the index is 4 and so
$(a^3)^4 = a^3 \times a^3 \times a^3 \times a^3 = a^{3+3+3+3} = a^{12}$
In this case, we *multiply* the powers

Rule 2	$(x^m)^n = x^{mn}$

It is very easy to confuse these two situations and is in fact the source of many slips and lost marks. Make sure you see the difference between $x^5 \times x^3 = x^8$ and $(x^5)^3 = x^{15}$.

Note:

1 Remember that although we don't write it in full usually, x actually means x^1 so that $x \times x^5 = x^1 \times x^5 = x^6$.

2 If the bases are not the same, the first rule rule does not apply, Be careful when multiplying something like 2×3^5: a common mistake is to say that it is the same as 6^5 – it isn't! (We would need to multiply $2^5 \times 3^5$ to get 6^5.)

Tip

When the expression is more complicated, work out the parts separately and then combine

For example $(a^4)^2 \times a \times (a^2)^2 = a^8 \times a^1 \times a^4 = a^{13}$

Here are some questions so that you can check that you are able to apply these rules.

Practice questions

1 (a) $x^2 \times x^5$ (b) $(x^2)^5$ (c) $x \times x^3$
(d) $x^2 \times x$ (e) $(x^3)^2$ (f) $x^2 \times (x^3)^2$
(g) $x \times (x^2)^3$ (h) $(x^3)^2 \times (x^4)^3$

Dividing terms

To simplify an expression like $a^7 \div a^4$, we can rewrite the division as a fraction and cancel the equal parts from the top and the bottom

$$a^7 \div a^2 = \frac{a^7}{a^2} = \frac{a \times a \times a \times a \times a \times a \times a}{a \times a} = a \times a \times a \times a \times a = a^5$$

In other words, when we divide two terms with the same base, we subtract the indices

Rule 3 $a^m \div a^n = a^{m-n}$

Example

Simplify

(a) $x^{10} \div (x^3)^2$

(b) $x^5 \div [\dfrac{x^8}{(x^2)^2}]$

Solution

(a) Working out the bracket first of all,

$x^{10} \div (x^3)^2 = x^{10} \div x^6 = x^{10-6} = x^4$

(b) Working out the square bracket first of all,

$$x^5 \div [\frac{x^8}{(x^2)^2}] = x^5 \div [\frac{x^8}{x^4}]$$

$$= x^5 \div x^4$$

$$= x^1 = x$$

Do some of these yourself.

Practice questions

2 Write the following in the form x^k:

(a) $x^7 \div x^4$ (b) $\dfrac{x^8}{x^3}$ (c) $x^4 \div x$

(d) $\dfrac{x^5}{x}$ (e) $x^6 \div (x^2)^2$ (f) $(x^3)^2 \div x$

(g) $\dfrac{x^4 \times x^3}{(x^2)^3}$ (h) $x^5 \div \dfrac{(x^4)^2}{x^7}$

Negative indices

If we try and apply the definition that xm means multiplying x by itself m times to the expression 2^{-3}, we don't seem to have anything meaningful: how can 2 be multiplied by itself a negative number of times? An alternative approach is to use the rule that $x^m \times x^n = x^{m+n}$ and see what happens if n is negative.

For example, taking m to be 5, n to be –3 and using 2 as the base, we have
$2^5 \times 2^{-3} = 2^2$. Then what value must we assign to 2^{-3} if the equation is to be true? We have $2^5 = 32$ and $2^2 = 4$ and so

$$32 \times 2^{-3} = 4 \quad \Rightarrow \quad 2^{-3} = \frac{4}{32} = \frac{1}{8} = \frac{1}{2^3}$$

So it looks as though things work out if we take 2^3 to be the same as $\frac{1}{2^3}$. In fact, whatever the base x and the index m, things work if we take x^{-m} to be the same as $\frac{1}{x^m}$

Rule 4 $\qquad x^{-m} = \frac{1}{x^m}$

In particular when m = 1, $x^{-1} = \frac{1}{x^1} = \frac{1}{x}$ is called the reciprocal of x:

$3^{-1} = \frac{1}{3}$ and $p^{-1} = \frac{1}{p}$ for example.

A more informal way of remembering this is that **the negative power turns things upside-down**: if they were at the top, in using the negative sign they go to the bottom, e.g.

$$x^{-8} = \frac{1}{x^8} \text{ and } 2x^{-1} = \frac{2}{x^1} = \frac{2}{x}$$

Similarly, anything with a negative index on the bottom comes to the top in losing the negative sign, e.g.

$$\frac{1}{x^{-3}} = x^3 \text{ and } \frac{1}{4x^{-1}} = \frac{x^1}{4} = \frac{x}{4}$$

You have to be careful to see which parts of an expression the power refers to.

For example $2x^{-1} = \frac{2}{x}$ and $(2x)^{-1} = \frac{1}{2x}$: $\frac{1}{4x^{-1}} = \frac{x}{4}$ and $\frac{1}{(4x)^{-1}} = 4x$

Example

Simplify

(a) $\dfrac{x^3}{x^{-2}}$

(b) $\dfrac{x^{-3}}{x^4}$

Solution

(a) $\dfrac{x^3}{x^{-2}} = x^3 \times x^2 = x^5$

(b) $\dfrac{x^{-3}}{x^4} = \dfrac{1}{x^3 \times x^4} = \dfrac{1}{x^7} = x^{-7}$

When working with numerical fractions raised to negative powers we remember that the negative sign just turns everything upside-down, e.g.

$3^{-2} = \dfrac{1}{3^2} = \dfrac{1}{9}$ $\qquad$ $\left(\dfrac{1}{2}\right)^{-3} = \left(\dfrac{2}{1}\right)^3 = 2^3 = 8$

$\left(\dfrac{3}{4}\right)^{-1} = \left(\dfrac{4}{3}\right)^1 = \dfrac{4}{3}$ $\qquad$ $\left(\dfrac{2}{3}\right)^{-3} = \left(\dfrac{3}{2}\right)^3 = \dfrac{27}{8}$

Now do the following questions.

Practice questions

3 Simplify and write the following in the form x^k:

(a) $\dfrac{x^8}{x^{-3}}$ (b) $\dfrac{x^{-2}}{x^3}$ (c) $\dfrac{x^{-1}}{x^2}$

(d) $\dfrac{x}{x^{-4}}$ (e) $x \times \dfrac{x^2}{x^{-1}}$ (f) $x^2 \div \dfrac{x^{-1}}{x^2}$

(g) $\left(\dfrac{x}{x^3}\right)^{-1}$ (h) $\left(\dfrac{1}{x^2}\right)^{-2}$

4 Rewrite the following as fractions or numbers:

(a) $\left(\dfrac{2}{3}\right)^{-1}$ (b) $\left(\dfrac{1}{3}\right)^{-2}$ (c) 4^{-2}

(d) $\left(\dfrac{3}{2}\right)^{-3}$ (e) $(-2)^{-1}$ (f) $-(3)^{-2}$

(g) $(-3)^{-2}$ (h) $\dfrac{-1}{3^{-1}}$

1.2.3 Multiplying expressions

When multiplying or dividing expressions involving constants, e.g. $3x^2 \times 4x^{-3}$, we deal with the constants first of all and then the variables

$$3x^2 \times 4x^{-3} = 12x^2 \times x^{-3} = 12x^{-1} = \frac{12}{x}$$

Tip

When the expressions are more complicated, begin with the brackets.

Example

(a) $4x^4 \times 2x^2$ (b) $5x^2 \times 2x^{-4}$ (c) $8x \div 4x^3$
(d) $(2x)^3 \times x$ (e) $(2xy^2)^3$

Solution

(a) $4x^4 \times 2x^2 = 8x^4 \times x^2 = 8x^6$ (b) $5x^2 \times 2x^{-4} = 10x^2 \times x^{-4} = 10x^{-2}$

(c) $8x \div 4x^3 = 2x \div x^3 = 2x^{1-3} = 2x^{-2}$

(d) $(2x)^3 \times x = 8x^3 \times x = 8x^4$ (e) $(2xy^2)^3 = 2^3 \times (xy^2)^3 = 8x^3y^6$

Have a go at some of these.

Practice questions

5 Simplify

(a) $3x^3 \times 4x^2$ (b) $10x^2 \div 2x$ (c) $2x^2 \times 5xy$ (d) $12x^3y \div 3xy$

(e) $(2x^{-2})^3 \times 4x^5$ (f) $(3x^{-2})^{-1} \div \dfrac{4}{x^2}$

1.2.4 Further indices

Index of zero

What happens if we have an index of zero? Taking a to be 2, m to be 5 and n to be zero in the rule $am \times an = am^{+}n$ gives $2^5 \times 2^0 = 2^{5+0} = 2^5$, i.e. $32 \times 2^0 = 32$. For this to be true, 2^0 must have the value of 1 and this is true in general, so that unless x is itself zero.

Rule 5	$x^0 = 1, \quad x \neq 0$

Fractional indices

We can further extend the values that the indices m and n can take if we allow them to be fractions and see where this leads to.

Taking the rule $(am)n = amn$ with $m = \frac{1}{2}$ and $n = 2$ gives

$$\left(a^{1/2}\right)^2 = a^{1/2 \times 2} = a^1 = a$$

and so when we square $a^{1/2}$ we end up with x. We already have a term for this, the <u>square-root</u> of x, written $\sqrt{x}$ and so $x^{1/2} = \sqrt{x}$
Similarly, $x^{1/3}$ is the cube-root of x, $\sqrt[3]{x}$. In general

Rule 6	$x^{1/p} = \sqrt[p]{x}$

Example

Evaluate the following, without using a calculator

(a) $27^{1/3}$ (b) $64^{-1/3}$ (c) $\dfrac{8^{-1/3}}{27}$ (d) $\left(\dfrac{8}{27}\right)^{-1/3}$

Solution

(a) $27^{1/3} = \sqrt[3]{27} = 3$ (b) $64^{-1/3} = \left(\dfrac{1}{64}\right)^{1/3} = \dfrac{1}{64^{1/3}} = \dfrac{1}{\sqrt[3]{64}} = \dfrac{1}{4}$

(c) $\dfrac{8^{-1/3}}{27} = \dfrac{1}{27\times 8^{1/3}} = \dfrac{1}{27\times 2} = \dfrac{1}{54}$ (d) $\left(\dfrac{8}{27}\right)^{-1/3} = \left(\dfrac{27}{8}\right)^{1/3} = \dfrac{27^{1/3}}{8^{1/3}} = \dfrac{3}{2}$

Now practise with the following questions.

Practice questions

6 Find the value, without a calculator, of

(a) $16^{\frac{1}{2}}$ (b) $16^{\frac{1}{4}}$ (c) $\left(\dfrac{9}{25}\right)^{\frac{1}{2}}$

(d) $\left(\dfrac{64}{27}\right)^{\frac{1}{3}}$ (e) $\left(\dfrac{9}{16}\right)^{-\frac{1}{2}}$ (f) $-4^{\frac{1}{2}}$

(g) $-8^{\frac{1}{3}}$ (h) $(-8)^{\frac{1}{3}}$ (i) $9^{-\frac{1}{2}}\times\left(\dfrac{27}{8}\right)^{\frac{1}{3}}$

(j) $\left(\dfrac{25}{4}\right)^{\frac{1}{2}}\times\left(\dfrac{125}{64}\right)^{-\frac{1}{3}}$

More complicated fractional indices

This is about as difficult as things can get! We use the fact that, for example, $\dfrac{3}{4}=\dfrac{1}{4}\times 3$ and the multiplication rule for indices to work out that

$$16^{\frac{3}{4}} = 16^{\frac{1}{4}\times 3} = \left(16^{\frac{1}{4}}\right)^3 = 2^3 = 8$$

We could actually have taken the multiplication the other way around, i.e. $16^{\frac{3}{4}} = 16^{3\times\frac{1}{4}} = \left(16^3\right)^{\frac{1}{4}} = \sqrt[4]{16^3} = \sqrt[4]{4096} = 8$

but this turns out to be more complicated. Usually, we apply the bottom part of the fraction first.

Keypoint

The general rule that you have to know here is:

$$a^{\frac{m}{n}} = \left(a^{\frac{1}{n}}\right)^m = \left(\sqrt[n]{a}\right)^m$$

$$= \left(a^m\right)^{\frac{1}{n}} = \sqrt[n]{a^m}$$

To work out values, it's usually easiest to do it in the following order: sign, bottom, top, e.g.

$$\left(\frac{27}{64}\right)^{-\frac{2}{3}} = \left(\frac{64}{27}\right)^{\frac{2}{3}} \qquad \text{[sign]}$$

$$= \left[\left(\frac{64}{27}\right)^{\frac{1}{3}}\right]^2 \qquad \text{[bottom]}$$

$$= \left[\frac{\sqrt[3]{64}}{\sqrt[3]{27}}\right]^2$$

$$= \left(\frac{4}{3}\right)^2$$

$$= \frac{16}{9} \qquad \text{[top]}$$

Watch the negative signs: $-8^{\frac{2}{3}} = -\left(8^{\frac{1}{3}}\right)^2 = -(2)^2 = -4$ is not the same as

$$(-8)^{\frac{2}{3}} = \left[(-8)^{\frac{1}{3}}\right]^2 = (-2)^2 = 4$$

Now try the following questions.

Practice questions

7 (a) $9^{\frac{3}{2}}$ (b) $8^{\frac{2}{3}}$ (c) $4^{-\frac{3}{2}}$

(d) $(-27)^{\frac{2}{3}}$ (e) $-27^{\frac{2}{3}}$ (f) $\left(\frac{8}{27}\right)^{\frac{2}{3}}$

(g) $\left(\frac{16}{9}\right)^{-\frac{3}{2}}$ (h) $\left(\frac{-27}{64}\right)^{-\frac{2}{3}}$ (i) $100^{\frac{3}{2}}$

(j) $1000^{-\frac{4}{3}} \times \left(\frac{100}{9}\right)^{\frac{3}{2}}$

1.2.5 Use in algebra

You will find that in later work you will need to combine expressions involving indices and fractions. Here are some typical examples

Example

Expand, leaving indices as fractions

(a) $(\sqrt{x}+1)^2$ (b) $x^{1/2}(x^2-4)$ (c) $(3\sqrt{x}-1)(x+1)$ (d) $(3\sqrt{x}+4)x^{-2}$

Solution

(a) $(\sqrt{x}+1)^2 = x+2\sqrt{x}+1 = x+2x^{1/2}+1$

(b) $x^{1/2}(x^2-4) = x^{1/2+2} - 4x^{1/2} = x^{5/2} - 4x^{1/2}$

(c) $(3\sqrt{x}-1)(x+1) = 3x\sqrt{x}+3\sqrt{x}-x-1 = 3x^{3/2}-x+3x^{1/2}-1$

(d) $(3\sqrt{x}+4)x^{-2} = (3x^{1/2}+4)x^{-2} = 3x^{-3/2}+4x^{-2}$

Example

Express the following in a form not involving fractions (except in the indices)

(a) $\dfrac{x^2+x^{1/2}}{x}$ (b) $\dfrac{(x-2)^2}{\sqrt{x}}$

Solution

(a) $\dfrac{x^2+x^{1/2}}{x} = (x^2+x^{1/2})x^{-1} = x^{2-1}+x^{1/2-1} = x+x^{-1/2}$

(b) $\dfrac{(x-2)^2}{\sqrt{x}} = (x^2-4x+4)\,x^{-1/2} = x^{3/2}-4x^{1/2}+4x^{-1/2}$

Practice questions

8 Expand the following, without using a calculator and leaving indices as fractions if appropriate

(a) $\sqrt{x}(x-1)$ (b) $(\sqrt{x}-1)^2$ (c) $(x^2+1)x^{-1/2}$ (d) $(x^{1/2}+1)(x^{-1/2}-1)$ (e) $x^2(\sqrt{x}-\frac{2}{\sqrt{x}})$ (f) $\left(\sqrt{x}-\frac{2}{\sqrt{x}}\right)^2$

1.2.6 Solving equations

Equations with a single term

We can use the multiplication rule of indices to solve simple equations such as $x^{\frac{1}{3}}=2$. If we raise both sides of this equation to the power of 3, we get

$$\left(x^{\frac{1}{3}}\right)^3=2^3$$

$$x^{\frac{1}{3}\times 3}=2^3$$

$x^1 = 2^3$, i.e. $x = 8$

You can see that you want to end up with a power of 1 for x. This suggests that when dealing with a fractional power, you raise both sides to the power upside-down:

$$x^{\frac{2}{3}}=\frac{4}{9}$$

We raise both sides to the power of $\frac{3}{2}$:

$$\left(x^{\frac{2}{3}}\right)^{\frac{3}{2}}=\left(\frac{4}{9}\right)^{\frac{3}{2}}$$

$$\Rightarrow x^{\frac{2}{3}\times\frac{3}{2}}=\left[\left(\frac{4}{9}\right)^{\frac{1}{2}}\right]^3$$

$$x^1 = \left(\frac{2}{3}\right)^3$$

$$\Rightarrow x = \frac{8}{27}$$

Tip

Be careful when you have an equation like $x^{\frac{1}{2}} = 4$. It's very easy to say that $x = 4^{\frac{1}{2}} = 2$ instead of the correct $x = 4^2 = 16$.

Any minus signs in the equation can be dealt with first.

Example

Solve $x^{-2} = 3$

Solution

Rewriting, $\dfrac{1}{x^2} = 3$

× by x^2 $\quad 1 = 3x^2$

÷ by 3 $\quad x^2 = \dfrac{1}{3}$

Taking square roots $x = \pm\dfrac{1}{\sqrt{3}}$

You *must* remember that ± sign when you take the square root of an equation!

Now do the following questions.

Practice questions

9 Solve the following equations:

(a) $x^3 = 8$ (b) $x^{\frac{1}{2}} = 3$ (c) $x^3 = -64$

(d) $x^{-3} = 27$ (e) $x^{-3} = -\dfrac{1}{8}$ (f) $x^{\frac{3}{2}} = 27$

(g) $x^{\frac{4}{3}} = 16$ (h) $x^{\frac{2}{3}} = 4$

Equations with unknown powers

At the moment, we can only solve a particular type of these equations, i.e. when both sides can be expressed as a power of the same number, so that eventually the equation looks like

$a^p = a^q$

We can then simply say that $p = q$ and find the solution from this. Let's have a look at a typical example of this kind.

Example

Solve the equation

$2 \times 4^{x-3} = 16^{2-x}$

Solution

1 Look for a number that all the numbers occurring in the equation are powers of.

Here we have 2, since $2 = 2^1$, $4 = 2^2$ and $16 = 2^4$.

2 Rewrite the equation as powers of this number (called the **base**):

$2 \times (2^2)^{x-3} = (2^4)^{2-x}$

3 Use the laws of indices to simplify both sides to the form of a number raised to a power:

$2^1 \times 2^{2(x-3)} = 2^{4(2-x)}$

$2^1 \times 2^{2x-6} = 2^{8-4x}$

$2^{2x-5} = 2^{8-4x}$

4 Equate the two powers and solve

$2x - 5 = 8 - 4x$

$6x = 13$

$$x = \frac{13}{6}$$

Here's another example.

Example

Solve the equation

$$\frac{3^{2x+1}}{27^{2-x}} = 9^{2x}$$

Solution

The base here is 3, with $27 = 3^3$ and $9 = 3^2$.

Rewriting in terms of the base:

$$\frac{3^{2x+1}}{\left(3^3\right)^{2-x}} = \left(3^2\right)^{2x}$$

$$\frac{3^{2x+1}}{3^{6-3x}} = 3^{4x}$$

Multiply both sides by 3^{6-3x}:

$3^{2x+1} = 3^{4x} \times 3^{6-3x}$

$= 3^{6+x}$

Equating powers:

$2x + 1 = 6 + x$

$\Rightarrow x = 5$.

Now try the following questions.

Practice questions

10 (a) $2 \times 4^{x+1} = 16^{2x}$

(b) $100^x = 1000$

(c) $3^{y^2+3} = 9^{2y}$

(d) $4^{2x} \times 8^{x-1} = 32$

(e) $2^{\frac{x}{2}} = 8 \times 2^{-\frac{x}{2}}$

(f) $3 \times 9^{x-1} = 27^x$

(g) $2^{3x} \times 4^{x-1} = 16$

(h) $4^{2x-1} = 16^{-\frac{1}{2}x}$

1.2.7 Surds

A **surd** is an **irrational root** of a **rational number**. We shall be looking at surds like $3 + 2\sqrt{5}$ and $\sqrt{7} - 4\sqrt{3}$, where the numbers inside the square roots are not perfect squares.

Note that although the square root of 4 is ±2, we define the number $\sqrt{4}$ to be the positive square root, i.e. +2. So $\sqrt{a}$ is, by definition, always positive.

Since surds are exact as they stand, it's frequently preferable to keep them as they are and learn how to manipulate them rather than using a decimal approximation such as $\sqrt{3} = 1.732$. If asked for an exact answer, you must use surds if appropriate, otherwise you will lose marks.

Be careful: $\sqrt{a}+\sqrt{b} \neq \sqrt{a+b}$!

For example $\sqrt{9}+\sqrt{16}=3+4=7$ whereas $\sqrt{9+16}=\sqrt{25}=5$

The rules for multiplying and dividing square roots are quite simple:

Keypoint

$$\sqrt{x}\times\sqrt{y}=\sqrt{xy}$$

$$\frac{\sqrt{x}}{\sqrt{y}}=\sqrt{\frac{x}{y}}$$

So, for example, $\sqrt{3} \times \sqrt{5} = \sqrt{15}$ and $\sqrt{21} \div \sqrt{3} = \sqrt{\frac{21}{3}} = \sqrt{7}$.

We can use the first rule to simplify the numbers inside the surd as far as possible, e.g.

$$\sqrt{50} = \sqrt{2\times 25} = \sqrt{2} \times \sqrt{25} = 5\sqrt{2}$$

and

$$\sqrt{12} = \sqrt{4\times 3} = \sqrt{4} \times \sqrt{3} = 2\sqrt{3}$$

As with algebraic expressions, we can collect like terms together, so that for example

$$3\sqrt{6} + 5\sqrt{6} = 8\sqrt{6} \text{ and } 10\sqrt{5} - 13\sqrt{5} = -3\sqrt{5}$$

Practise by doing the following questions.

Practice questions

11 Simplify the following

(a) $\sqrt{18}$

(b) $\sqrt{98}$

(c) $\sqrt{80}$

(d) $\sqrt{200}$

(e) $\sqrt{32}$

(f) $\sqrt{75}$

(g) $\dfrac{\sqrt{35}}{\sqrt{7}}$ (h) $\dfrac{\sqrt{50}}{\sqrt{8}}$

(i) $\dfrac{\sqrt{128}}{\sqrt{4}}$ (j) $\dfrac{\sqrt{1000}}{\sqrt{75}}$

12 Simplify and collect like terms:

(a) $\sqrt{18} + \sqrt{8}$ (b) $2\sqrt{12} + 3\sqrt{75}$

(c) $3\sqrt{5} + \sqrt{80}$ (d) $\sqrt{50} - \sqrt{32}$

(e) $\sqrt{200} - 2\sqrt{98}$ (f) $2\sqrt{80} + 3\sqrt{125} - 4\sqrt{20}$

13 Expand and simplify as far as possible

(a) $(\sqrt{3} - \sqrt{2})(\sqrt{3} + \sqrt{2})$ (b) $(\sqrt{3} - 1)^2$

(c) $(2\sqrt{5} - \sqrt{3})(\sqrt{5} + 2\sqrt{3})$ (d) $(\sqrt{8} + \sqrt{2})^2$

(e) $(\sqrt{x} + \sqrt{y})(\sqrt{x} - \sqrt{y})$ (f) $\left(\sqrt{pq^3} - \sqrt{p^3q}\right)^2$

1.2.8 Rationalising the denominator

We prefer to have any surds in the numerator rather than the denominator if they occur in a fraction: the process of eliminating the surds on the bottom of a fraction is called **rationalising the denominator**. In the simplest case, we have a single surd as denominator, something like

$$\frac{6}{\sqrt{2}}$$

We then multiply the top and bottom of the fraction by √2. By doing this, we square the denominator and clear it of surds:

$$\frac{6}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} = \frac{6\sqrt{2}}{2} = 3\sqrt{2}$$

When the denominator consists of a mixture of numbers and roots, e.g. $2 - \sqrt{3}$, or $\sqrt{5} + 2\sqrt{2}$, we use a different method which relies on the fact that $(a + b)(a - b) = a^2 - b^2$. So, if we multiply either of the above expressions by the same expression with an opposite sign in the middle, we end up with the difference of two squares, which eliminates the surds:

$(2 - \sqrt{3})(2 + \sqrt{3}) \quad = 2^2 - (\sqrt{3})^2$

$= 4 - 3 = 1$

$(\sqrt{5} + 2\sqrt{2})(\sqrt{5} - 2\sqrt{2}) \quad = (\sqrt{5})^2 - (2\sqrt{2})^2$

$= 5 - 8 = -3$

Be careful when you're squaring mixtures of numbers and surds, e.g.

$(3\sqrt{2})^2 = 3\sqrt{2} \times 3\sqrt{2} = 9 \times 2 = 18$

Let's see how this method of multiplying by the opposite expression clears the denominator of surds.

Example

(a) Express $\dfrac{14}{3-\sqrt{2}}$ in the form $a + b\sqrt{2}$

(b) Express $\dfrac{5}{2\sqrt{2}-\sqrt{7}}$ in the form $c\sqrt{2} + d\sqrt{7}$

Solution

(a) Multiply the top and bottom of the fraction by the opposite of $3 - \sqrt{2}$, i.e. $3 + \sqrt{2}$:

$$\frac{14}{3-\sqrt{2}} \times \frac{3+\sqrt{2}}{3+\sqrt{2}} = \frac{14\left(3+\sqrt{2^2}\right)}{3^2-\sqrt{2^2}} = \frac{14\left(3+\sqrt{2}\right)}{9-2}$$

$$= \frac{14\left(3+\sqrt{2}\right)}{7} = 2\left(3+\sqrt{2}\right)$$

$$= 6 + 2\sqrt{2}$$

(b) This time, multiply the top and bottom by $2\sqrt{2} + \sqrt{7}$:

$$\frac{5}{2\sqrt{2}-\sqrt{7}} \times \frac{2\sqrt{2}+\sqrt{7}}{2\sqrt{2}+\sqrt{7}} = \frac{5\left(2\sqrt{2}+\sqrt{7}\right)}{\left(2\sqrt{2}\right)^2-\left(\sqrt{7}\right)^2} = \frac{5\left(2\sqrt{2}+\sqrt{7}\right)}{8-7}$$

$$= 5\left(2\sqrt{2}+\sqrt{7}\right)$$

$$= 10\sqrt{2} + 5\sqrt{7}$$

Practise this with the following questions.

Practice questions

14 Rationalise the denominator and simplify where possible:

(a) $\frac{12}{\sqrt{3}}$ (b) $\frac{20}{\sqrt{5}}$ (c) $\frac{16}{\sqrt{8}}$

(d) $\frac{5}{\sqrt{2}}$ (e) $\frac{3\sqrt{2}}{\sqrt{3}}$ (f) $\frac{4\sqrt{7}}{\sqrt{2}}$

(g) $\frac{9\sqrt{50}}{2\sqrt{3}}$ (h) $\frac{7\sqrt{18}}{3\sqrt{35}}$

15 Rationalise the denominator:

(a) $\frac{1}{2-\sqrt{3}}$ (b) $\frac{3}{4-\sqrt{7}}$ (c) $\frac{2\sqrt{2}}{\sqrt{2}+1}$

(d) $\frac{12}{3-\sqrt{3}}$ (e) $\frac{6}{\sqrt{5}-\sqrt{2}}$ (f) $\frac{2+\sqrt{2}}{2-\sqrt{2}}$

(g) $\frac{8\sqrt{5}}{3-\sqrt{5}}$ (h) $\frac{5\sqrt{2}-10\sqrt{7}}{\sqrt{7}-\sqrt{2}}$

16 Solve the equation

$$x\sqrt{8}-11=\frac{3x}{\sqrt{2}}$$

giving your answer in the form $k\sqrt{2}$, where k is an integer.

17 Express each of the following in the form $p + q\sqrt{3}$:

(a) $(2 + \sqrt{3})(5 - 2\sqrt{3})$

(b) $\frac{26}{4-\sqrt{3}}$

Summary exercise

1 (a) Given that $8 = 2^k$, write down the value of k.

(b) Given that $4^x = 8^{2-x}$, find the value of x.

2 (a) Given that $(2 + \sqrt{7})(4 - \sqrt{7}) = a + b\sqrt{7}$, where a and b are integers, find the value of a and the value of b.

(b) Given that $\dfrac{2+\sqrt{7}}{4+\sqrt{7}} = c + d\sqrt{7}$, where c and d are rational numbers,

find the value of c and the value of d.

3 Simplify

$$\frac{\left(2\sqrt{3}-\sqrt{2}\right)^2}{\sqrt{6}-2}$$

expressing your answer in surd form.

4 (a) Write each of the following as a power of 3:

(i) $\dfrac{1}{27}$

(ii) 9^x

(b) Hence solve the equation $9^x \times 3^{1-x} = \dfrac{1}{27}$

5 It is given that $4^x = 8^{2x+1}$. By expressing each side of this equation as a power of 2, or otherwise, find the value of x.

6 It is given that $a^p = 5$ and $a^q = 9$. In each of the following cases, determine the numerical value of the given expression:

(a) a^{p+q} (b) $2a^{-p}$ (c) $a^{2p-\frac{1}{2}q}$

7 Without a calculator, find the value of

(a) $\dfrac{3}{\sqrt{2}-1}-\dfrac{6}{\sqrt{2}}$ (b) $\dfrac{1}{3-\sqrt{5}}+\dfrac{1}{3+\sqrt{5}}$

8 Solve the equations

(a) $x^{\frac{2}{3}} = 8x$ (b) $2 \times 4^{x+1} = 8^x$

9 When $x = \dfrac{-1}{2}$, find the value of

(a) $\dfrac{1}{3x}$ (b) $\dfrac{1}{3x^{-2}}$

10 (a) Show that the substitution $y = 2^x$ transforms the equation

$2(2^x) + 2^{-x} = 3$

into the quadratic equation

$2y^2 - 3y + 1 = 0$.

(b) Hence find the values of x that satisfy the equation

$2(2^x) + 2^{-x} = 3$.

11 Without a calculator, find the values of

(a) $\left(5^3\right)^{\frac{2}{3}}$ (b) $4^{-\frac{3}{2}}$ (c) $8^{\frac{4}{3}} \div 8^{\frac{2}{3}}$

12 (a) By substituting $X = 4^x$ and $Y = 3^y$, express the following simultaneous equations in terms of X and Y:

$4^{x+1} - 3^y = 1$

$4^x + 3^y = 1.5$

(b) Solve the new equations to obtain the values of X and Y.

(c) Hence find the values of x and y.

13 Solve the simultaneous equations

$\frac{125^x}{25^y} = 625$ and $2 \times 4^x = 32^y$

14 By means of the substitution $y = 2^x$, find the value of x such that $2^{x+2} - 2 = 7 \times 2^{x-1}$

15 Given that $b = a^6$,

(a) (i) Express $b^{\frac{1}{8}}$ as a power of a.

(ii) Given that

$c = \frac{\left(a^{\frac{3}{4}} + 5b^{\frac{1}{8}}\right)}{a^{\frac{1}{2}}}$ and $b = a^6$

write c in the form ka^p, stating the values of the constants k and p.

(b) Simplify $\left(4a^{\frac{1}{2}}b\right)^2 \div \left(2a^{-1}b^2\right)$.

16 By using the substitution $y = x^{\frac{1}{3}}$, solve the equation

$x^{\frac{2}{3}} - 5x^{\frac{1}{3}} + 6 = 0$.

17

(a) (i) Simplify

$\sqrt{50} + \sqrt{18}$

giving your answer in the form $a\sqrt{2}$ where a is an integer.

(ii) Hence, or otherwise, simplify

$$\frac{12\sqrt{3}}{\sqrt{50} - \sqrt{18}}$$

giving your answer in the form $b\sqrt{c}$ where b and c are integers and $b \neq 1$

(b) Solve the equation

$$10 + x\sqrt{8} = \frac{6x}{\sqrt{2}}$$

Give your answer in the form

(c) (i) Write $\sqrt{80}$ in the form $c\sqrt{5}$, where c is a positive constant.

(ii) A rectangle R has a length of $(1 + \sqrt{5})$ cm and an area of $\sqrt{80}$ cm^2. Calculate the width of R in cm. Express your answer in the form $p + q\sqrt{5}$, where p and q are integers to be found.

1.2.9 Summary of key points

In this topic we have seen that

$a^m \times a^n = a^{m+n}$ and $(a^m)^n = a^{mn}$

$a^m \div a^n = a^{m-n}$ and $\left(a^m\right)^{\frac{1}{n}} = a^{\frac{m}{n}}$

$$a^{-n} = \frac{1}{a^n}$$

$$a^{\frac{1}{n}} = \sqrt[n]{a}$$

$$a^{\frac{m}{n}} = \sqrt[n]{a^m}$$

- we can evaluate numbers raised to powers by applying the rules of indices sequentially
- we can solve certain equations by rewriting all the terms as powers of a common base, usually 2 or 3
- we can change certain equations involving unknown powers into quadratic equations using an appropriate substitution
- to add or multiply surds, we collect like terms together
- $\sqrt{a} \times \sqrt{b} = \sqrt{ab}$; $\quad \dfrac{\sqrt{a}}{\sqrt{b}} = \sqrt{\dfrac{a}{b}}$
- to rationalise $\dfrac{p}{\sqrt{q}}$, we multiply the top and bottom by $\sqrt{q}$
- to rationalise $\dfrac{p}{\sqrt{q}+r}$, we multiply the top and bottom by $\sqrt{q} - r$

Answers to practice questions

1	(a) x^7	(b) x^{10}	(c) x^4	(d) x^3
	(e) x^6	(f) x^8	(g) x^7	(h) x^{18}
2	(a) x^3	(b) x^5	(c) x^3	(d) x^4
	(e) x^2	(f) x^5	(g) x	(h) x^4
3	(a) x^{11}	(b) x^{-5}	(c) x^{-3}	(d) x^5
	(e) x^4	(f) x^5	(g) x^2	(h) x^4
4	(a) $\frac{3}{2}$	(b) 9	(c) $\frac{1}{16}$	(d) $\frac{8}{27}$
	(e) $-\frac{1}{2}$	(f) $-\frac{1}{9}$	(g) $\frac{1}{9}$	(h) -3

5 (a) $12x^5$ (b) $5x$ (c) $10x^3y$ (d) $4x^2$ (e) $32x^{-1}$ (f) $1/12x^4$

6	(a) 4	(b) 2	(c) $\frac{3}{5}$	(d) $\frac{4}{3}$
	(e) $\frac{4}{3}$	(f) -2	(g) -2	(h) -2
	(i) $\frac{1}{2}$	(j) 2		
7	(a) 27	(b) 4	(c) $\frac{1}{8}$	(d) 9
	(e) -9	(f) $\frac{4}{9}$	(g) $\frac{27}{64}$	(h) $\frac{16}{9}$
	(i) 1000	j) $\frac{1}{270}$		

8 (a) $x^{3/2} - x^{1/2}$ (b) $x - 2x^{1/2} + 1$ (c) $x^{3/2} - x^{-1/2}$ (d) $x^{-1/2} - x^{1/2}$
(e) $x^{5/2} - 2x^{3/2}$ (f) $x - 4 + 4x^{-1}$

9	(a) 2	(b) 9	(c) -4	(d) $\frac{1}{3}$
	(e) -2	(f) 9	(g) ± 8	(h) ± 8
10	(a) $\frac{1}{2}$	(b) $\frac{3}{2}$	(c) 1, 3	(d) $\frac{8}{7}$
	(e) 3	(f) -1	(g) $\frac{6}{5}$	(h) $\frac{1}{3}$
11	(a) $3\sqrt{2}$	(b) $7\sqrt{2}$	(c) $4\sqrt{5}$	(d) $10\sqrt{2}$
	(e) $4\sqrt{2}$	(f) $5\sqrt{3}$	(g) $\sqrt{5}$	(h) $\frac{5}{2}$

(i) $4\sqrt{2}$ (j) $\sqrt{\dfrac{40}{3}}$

12 (a) $5\sqrt{2}$ (b) $19\sqrt{3}$ (c) $7\sqrt{5}$ (d) $\sqrt{2}$
(e) $-4\sqrt{2}$ (f) $15\sqrt{5}$

13 (a) 1 (b) $4 - 2\sqrt{3}$ (c) $4 + 3\sqrt{15}$ (d) 18
(e) $x - y$ (f) $pq(q^2 - 2pq + p^2) \equiv pq(q - p)^2$

14 (a) $4\sqrt{3}$ (b) $4\sqrt{5}$ (c) $2\sqrt{8} = 4\sqrt{2}$ (d) $\dfrac{5}{2}\sqrt{2}$
(e) $\sqrt{6}$ (f) $2\sqrt{14}$ (g) $\dfrac{15}{2}\sqrt{6}$ (h) $\dfrac{1}{5}\sqrt{70}$

15 (a) $2 + \sqrt{3}$ (b) $\dfrac{4+\sqrt{7}}{3}$
(c) $2\sqrt{2}(\sqrt{2} - 1) = 4 - 2\sqrt{2}$ (d) $2(3 + \sqrt{3})$
(e) $2(\sqrt{5} + \sqrt{2})$ (f) $3 + 2\sqrt{2}$
(g) $2\sqrt{5}(3 + \sqrt{5}) = 6\sqrt{5} + 10$ (h) $-12 - \sqrt{14}$

16 $11\sqrt{2}$

17 (a) $4 + \sqrt{3}$ (b) $8 + 2\sqrt{3}$

What next?

We hope this sample has helped you to decide whether this course is right for you.

If you have any further questions, please do not hesitate to contact us using the details below.

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